

Structured Parseval Frames in Hilbert C^* -modules

Wu Jing, Deguang Han, and Ram N. Mohapatra

ABSTRACT. We investigate the structured frames for Hilbert C^* -modules. In the case that the underlying C^* -algebra is a commutative W^* -algebra, we prove that the set of the Parseval frame generators for a unitary operator group can be parameterized by the set of all the unitary operators in the double commutant of the group. Similar result holds for the set of all the general frame generators where the unitary operators are replaced by invertible and adjointable operators. Consequently, the set of all the Parseval frame generators is path-connected. We also obtain the existence and uniqueness results for the best Parseval multi-frame approximations for multi-frame generators of unitary operator groups on Hilbert C^* -modules when the underlying C^* -algebra is commutative.

1. Introduction

Frames (modular frames) for Hilbert C^* -modules were introduced by Frank and Larson and some basic properties were also investigated in a series of their papers [2, 3, 4]. It should be remarked that although (at the first glance) some of the definitions and result statements of modular frames may appear look like similar to their Hilbert space frame counterparts, these are not simple generalizations of the Hilbert space frames due to the complexity of the Hilbert C^* -module structures and to the fact that many useful techniques in Hilbert spaces are either not available or not known in Hilbert C^* -modules. For example, it is well-known that that every Hilbert space has an orthonormal basis

1991 *Mathematics Subject Classification.* Primary 46L99; Secondary 42C15, 46H25.

Key words and phrases. Frames, Parseval frame vectors, mutil-frame approximations, unitary groups, Hilbert C^* -modules.

which can be simply obtained by applying the Gram-Schmidt orthonormalization process to a linearly independent generating subset of the Hilbert space. However, it is well-known that not every Hilbert C^* -module has an “orthonormal basis”. This makes frames particularly relevant to Hilbert C^* -modules. Remarkably every countably generated Hilbert C^* -module admits a (countable) frame. It requires a very deep Hilbert C^* -module result (Kasparov’s Stabilization Theorem) to prove this fact which is a trivial fact in Hilbert space setting (cf [4, 18]. In fact, it is still an interesting question whether there exists an alternative proof for this fact without using the Kasparov’s Stabilization Theorem. Another still open problem is whether every uncountably generated Hilbert C^* -module admits a (uncountably indexed) Parseval frame (again, a trivial fact for Hilbert spaces). Equivalently, for every Hilbert C^* -module over a unital C^* -algebra \mathcal{A} , does there exist an isometric embedding into a standard Hilbert C^* -module $l^2(\mathcal{A}, I)$ as an orthogonal direct summand for some index set I ? (see [4]).

In recent years, there have been growing evidence indicating that modular frames are also closely related to some other areas of research such as the area of wavelet frame constructions (cf. [14, 15, 16, 22]). Considering the fact that the theory and applications of structured frames (such as Gabor frame, wavelet frames and frames induced from group unitary representations) for Hilbert spaces have been the main focus of the Hilbert space frame theory, we believe that structured modular frames may well be suitable for some applications either in theoretical or applied nature. The purpose of this paper is to initiate the study of structured modular frames. It is reasonable that we should first take a close look at those existed results for structured Hilbert space frames and make an effort to check whether they are still valid for structured modular frames. The two results (Theorems 3.2 and 4.3) presented in this paper are generalizations of the corresponding Hilbert space frame results obtained in [9] and [6]. Theorem 3.2 states that all the Parseval frame generators for a unitary group can be parameterized in terms of the unitary elements in the double commutant of the group under the commutativity condition on the underlying C^* -algebras. This is slightly different from the Hilbert space setting since the double commutant theorem for von Neumann algebras is not always available for the Hilbert C^* -module setting. For the similar reason, the “finiteness” of the involved “commutant” algebras need to be verified, and each step needs to be carefully checked to make sure it is valid in the C^* -algebra context. Theorem 4.3 deals with the best approximations of modular frame generators by Parseval frame generators. The difficulty arises when comes to compare two positive

elements in the underlying C^* -algebras which is not an issue in the scalar case. We are not able to prove these results when the underlying C^* -algebras are non-commutative.

2. Preliminaries

This section contains some basic definitions about Hilbert C^* -modules and some simple properties for Hilbert C^* -module frames that will be needed in the next two sections. Let \mathcal{A} be a C^* -algebra and \mathcal{H} be a (left) \mathcal{A} -module. Suppose that the linear structures given on \mathcal{A} and \mathcal{H} are compatible, i.e. $\lambda(ax) = a(\lambda x)$ for every $\lambda \in \mathbb{C}$, $a \in \mathcal{A}$ and $x \in \mathcal{H}$. If there exists a mapping $\langle \cdot, \cdot \rangle : \mathcal{H} \times \mathcal{H} \rightarrow \mathcal{A}$ with the properties

- (1) $\langle x, x \rangle \geq 0$ for every $x \in \mathcal{H}$,
- (2) $\langle x, x \rangle = 0$ if and only if $x = 0$,
- (3) $\langle x, y \rangle = \langle y, x \rangle^*$ for every $x, y \in \mathcal{H}$,
- (4) $\langle ax, y \rangle = a\langle x, y \rangle$ for every $a \in \mathcal{A}$, every $x, y \in \mathcal{H}$,
- (5) $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$ for every $x, y, z \in \mathcal{H}$.

Then the pair $\{\mathcal{H}, \langle \cdot, \cdot \rangle\}$ is called a (left)-pre-Hilbert \mathcal{A} -module. The map $\langle \cdot, \cdot \rangle$ is said to be an \mathcal{A} -valued inner product. If the pre-Hilbert \mathcal{A} -module $\{\mathcal{H}, \langle \cdot, \cdot \rangle\}$ is complete with respect to the norm $\|x\| = \|\langle x, x \rangle\|^{\frac{1}{2}}$ then it is called a *Hilbert \mathcal{A} -module*.

A Hilbert \mathcal{A} -module \mathcal{H} is (algebraically) *finitely generated* if there exists a finite set $\{x_1, \dots, x_n\} \subseteq \mathcal{H}$ such that every element $x \in \mathcal{H}$ can be expressed as an \mathcal{A} -linear combination $x = \sum_{i=1}^n a_i x_i$, $a_i \in \mathcal{A}$. A Hilbert \mathcal{A} -module is *countably generated* if there exists a countable set of generator.

It should be mentioned that by no means all results of Hilbert space theory can be simply generalized to the situation of Hilbert C^* -modules. First of all, the analogue of the Riesz representation theorem for bounded \mathcal{A} -linear mapping is not valid for \mathcal{H} . Secondly, the bounded \mathcal{A} -linear operator on \mathcal{H} may not have an adjoint operator. Thirdly, the Hilbert \mathcal{A} -submodule \mathcal{I} of the Hilbert \mathcal{A} -module \mathcal{H} is not a direct summand. Let \mathcal{H} be a Hilbert \mathcal{A} -module over a unital C^* -algebra \mathcal{A} . The set of all bounded \mathcal{A} -linear operators on \mathcal{H} is denoted by $End_{\mathcal{A}}(\mathcal{H})$, and the set of all adjointable bounded \mathcal{A} -linear operators on \mathcal{H} is denoted by $End_{\mathcal{A}}^*(\mathcal{H})$.

A C^* -algebra \mathcal{M} is called a *W^* -algebra* if it is a dual space as a Banach space, i.e. if there exists a Banach space \mathcal{M}_* such that $(\mathcal{M}_*)^* = \mathcal{M}$. We also call \mathcal{M}_* the predual of \mathcal{M} . It should mention here that $End_{\mathcal{A}}^*(l^2(\mathcal{A}))$ is not a W^* -algebra in general. A W^* -algebra \mathcal{M} is said to be *finite* if its identity is finite. Equivalently, \mathcal{M} is finite if and only if every isometry in \mathcal{M} is unitary.

DEFINITION 2.1. Let \mathcal{A} be a unital C^* -algebra and \mathbb{J} be a finite or countable index set. A sequence $\{x_j\}_{j \in \mathbb{J}}$ of elements in a Hilbert \mathcal{A} -module \mathcal{H} is said to be a (standard) frame if there exist two constants $C, D > 0$ such that

$$(2.1) \quad C \cdot \langle x, x \rangle \leq \sum_{j \in \mathbb{J}} \langle x, x_j \rangle \langle x_j, x \rangle \leq D \cdot \langle x, x \rangle$$

for every $x \in \mathcal{H}$, where the sum in the middle of the inequality is convergent in norm. The optimal constants (i.e. maximal for C and minimal for D) are called frame bounds.

The frame $\{x_j\}_{j \in \mathbb{J}}$ is said to be tight frame if $C = D$, and said to be Parseval if $C = D = 1$.

Note that not every Hilbert C^* -module has an orthonormal basis. Though any countably generated Hilbert C^* -module admits a frame, there are countably generated Hilbert C^* -modules that contain no orthonormal basis even no orthogonal Riesz basis (see Example 3.4 in [4]).

The main property of frames for Hilbert spaces is the existence of the reconstruction formula that allows a simple standard decomposition of every element of the spaces with respect to the frame. For standard frames we have the following reconstruction formula.

THEOREM 2.2. ([4]) Let $\{x_j\}_{j \in \mathbb{J}}$ be a standard frame in a finitely or countably generated Hilbert \mathcal{A} -module \mathcal{H} over a unital C^* -algebra \mathcal{A} . Then there exists a unique operator $S \in \text{End}_{\mathcal{A}}^*(\mathcal{H})$ such that

$$x = \sum_{j \in \mathbb{J}} \langle x, S(x_j) \rangle x_j$$

for every $x \in \mathcal{H}$. The operator can be explicitly given by the formula $S = T^*T$ for any adjointable invertible bounded operator T mapping \mathcal{H} onto some other Hilbert \mathcal{A} -module \mathcal{K} and realizing $\{T(x_j) : j \in \mathbb{J}\}$ to be a standard Parseval frame in \mathcal{K} .

Let $\mathcal{S} \subseteq \text{End}_{\mathcal{A}}(\mathcal{H})$, we denote its commutant $\{A \in \text{End}_{\mathcal{A}}(\mathcal{H}) : AS = SA, S \in \mathcal{S}\}$ by \mathcal{S}' . For a non-empty set \mathcal{U} and a unital C^* -algebra \mathcal{A} , let $l_{\mathcal{U}}^2(\mathcal{A})$ be the Hilbert \mathcal{A} -module defined by

$$l_{\mathcal{U}}^2(\mathcal{A}) = \{\{a_U\}_{U \in \mathcal{U}} \subseteq \mathcal{A} : \sum_{U \in \mathcal{U}} a_U a_U^* \text{ converges in } \|\cdot\|\}.$$

Let $\{\chi_U\}_{U \in \mathcal{U}}$ denote the standard orthonormal basis of $l_{\mathcal{U}}^2(\mathcal{A})$, where χ_U takes value $1_{\mathcal{A}}$ at U and $0_{\mathcal{A}}$ at everywhere else. In the case when \mathcal{U} is a group, we define for each $U \in \mathcal{U}$,

$$L_U \chi_V = \chi_{UV} \quad \text{and} \quad R_U \chi_V = \chi_{VU^{-1}}.$$

Note that $L_U^{-1} = L_U^* = L_{U^*}$ and $R_U^{-1} = R_U^* = R_{U^*}$. Here L and R are the *left* and *right regular representations* of \mathcal{U} .

Let \mathcal{H} be a Hilbert \mathcal{A} -module over a unital C^* -algebra \mathcal{A} . A vector ψ in \mathcal{H} is called a *wandering vector* for a unitary group \mathcal{U} on \mathcal{H} if $\mathcal{U}\psi = \{U\psi : U \in \mathcal{U}\}$ is an orthonormal set. If $\mathcal{U}\psi$ is an orthonormal basis for \mathcal{H} , then ψ is called a *complete wandering vector* for \mathcal{U} . Similarly, a vector ψ is called a *Parseval frame vector* (resp. *frame vector with bounds C and D* , or *Bessel sequence vector with bound D*) for a unitary group \mathcal{U} if $\mathcal{U}x$ forms a Parseval frame (resp. frame with bounds C and D , or Bessel sequence with bound D) for $\overline{\text{span}}(\mathcal{U}x)$. Moreover, x is called a *complete Parseval frame vector* (resp. *complete frame vector with bounds C and D* , or *complete Bessel sequence with bound D*) when $\mathcal{U}x$ is a Parseval frame (resp. frame with bounds C and D , or Bessel sequence with bound D) for \mathcal{H} .

The following simple lemma will be used in the proof of Theorem 3.2.

LEMMA 2.3. *Let \mathcal{G} be a unitary group on a finitely or countably generated Hilbert \mathcal{A} -module \mathcal{H} over a unital C^* -algebra \mathcal{A} . If \mathcal{G} admits a complete Parseval frame vector η , then \mathcal{G} is unitarily equivalent to $\{L_U|_{\mathcal{K}} : U \in \mathcal{U}\}$, where $\mathcal{K} = T(\mathcal{H})$ and $T : \mathcal{H} \rightarrow l_{\mathcal{G}}^2(\mathcal{A})$ be the analysis operator defined by $T(x) = \sum_{U \in \mathcal{G}} \langle x, U\eta \rangle \chi_U$.*

PROOF. It is easy to check that T is an adjointable isometry. By Theorem 15.3.5 and Theorem 15.3.8 in [21], we have that

$$l_{\mathcal{G}}^2(\mathcal{A}) = (T(\mathcal{H}))^\perp \oplus T(\mathcal{H}).$$

Hence we have the orthogonal projection P from $l_{\mathcal{G}}^2(\mathcal{A})$ onto $T(\mathcal{H})$.

For each $V \in \mathcal{G}$, we have

$$\begin{aligned} L_U T(V\eta) &= L_U \left(\sum_{W \in \mathcal{G}} \langle V\eta, W\eta \rangle \chi_W \right) = \sum_{W \in \mathcal{G}} \langle V\eta, W\eta \rangle \chi_{UW} \\ &= \sum_{W \in \mathcal{G}} \langle UV\eta, UW\eta \rangle \chi_{UW} = \sum_{W \in \mathcal{G}} \langle UV\eta, W\eta \rangle \chi_W \\ &= TU(V\eta). \end{aligned}$$

Thus $L_U T = TU$. □

We remark that the orthogonal projection from $l_{\mathcal{G}}^2(\mathcal{A})$ onto $T(\mathcal{H})$ is in the commutant of $\{L_U : U \in \mathcal{G}\}$ and satisfies $T\eta = P\chi_I$. This also implies the so-called *dilation property* meaning that there exists a Hilbert \mathcal{A} -module $\tilde{\mathcal{H}} \supseteq \mathcal{H}$ and a unitary group $\tilde{\mathcal{G}}$ on $\tilde{\mathcal{H}}$ such that $\tilde{\mathcal{G}}$ has complete wandering vectors in $\tilde{\mathcal{H}}$, \mathcal{H} is an invariant subspace of $\tilde{\mathcal{G}}$ such

that $\tilde{\mathcal{G}}|_{\mathcal{H}} = \mathcal{G}$, and the map $G \mapsto G|_{\mathcal{H}}$ is a group isomorphism from $\tilde{\mathcal{G}}$ onto \mathcal{G} .

3. Frame Vector Parameterizations

In [1] the set of all wandering vectors for a unitary group was parameterized by the set of unitary operators in its commutant. However, unlike the wandering vector case, it was shown in [9] that the set of all the Parseval frame vectors for a unitary group can not be parameterized by the set of all the unitary operators in the commutant of the unitary group. This means that the Parseval frame vectors for a representation of a countable group are not necessarily unitarily equivalent. However, this set can be parameterized by the set of all the unitary operators in the von Neumann algebra generated by the representation ([6, 9]). This turns out to be a very useful result in Gabor analysis (cf. [6, 8]). Although it remains a question whether this result is still valid in the Hilbert C^* -module setting, we will prove in this section that this result holds when the underlying C^* -algebra is a commutative W^* -algebra. Even in this commutative case, a lot more extra work and care are needed in order to prove this generalization.

LEMMA 3.1. *Let \mathcal{G} be a unitary group on a Hilbert \mathcal{A} -module over a commutative unital C^* -algebra \mathcal{A} , then*

$$\mathcal{M} = \mathcal{N}' = \{R_U : U \in \mathcal{G}\}' \quad \text{and} \quad \mathcal{N} = \mathcal{M}' = \{L_U : U \in \mathcal{G}\}',$$

where $\mathcal{M} = \{L_U : U \in \mathcal{G}\}''$ and $\mathcal{N} = \{R_U : U \in \mathcal{G}\}''$.

PROOF. Note that $R_U L_V = L_V R_U$ holds for any $U, V \in \mathcal{G}$. Therefore to prove this lemma it suffices to show that $TS = ST$ for arbitrary $T \in \mathcal{M}'$ and $S \in \mathcal{N}'$.

Suppose that

$$T\chi_I = \sum_{U \in \mathcal{G}} a_U \chi_U \quad \text{and} \quad S\chi_I = \sum_{U \in \mathcal{G}} b_U \chi_U$$

for some $a_U, b_U \in \mathcal{A}$.

Now for any $V \in \mathcal{G}$, on one hand, we have

$$\begin{aligned} ST\chi_V &= STL_V\chi_I = SL_V T\chi_I \\ &= SL_V \left(\sum_{U \in \mathcal{G}} a_U \chi_U \right) = S \left(\sum_{U \in \mathcal{G}} a_U \chi_{VU} \right) \\ &= S \left(\sum_{U \in \mathcal{G}} a_U R_{(VU)^{-1}} \chi_I \right) = \sum_{U \in \mathcal{G}} a_U R_{(VU)^{-1}} S\chi_I \\ &= \sum_{U \in \mathcal{G}} a_U R_{(VU)^{-1}} \left(\sum_{W \in \mathcal{G}} b_W \chi_W \right) = \sum_{U, W \in \mathcal{G}} a_U b_W \chi_{WVU}. \end{aligned}$$

On the other hand

$$\begin{aligned}
TS\chi_V &= TSR_{V^{-1}}\chi_I = TR_{V^{-1}}S\chi_I \\
&= TR_{V^{-1}}\left(\sum_{W \in \mathcal{G}} b_W \chi_W\right) = T\left(\sum_{W \in \mathcal{G}} b_W \chi_{WV}\right) \\
&= T\left(\sum_{W \in \mathcal{G}} b_W L_{WV}\chi_I\right) = \sum_{W \in \mathcal{G}} b_W L_{WV}T\chi_I \\
&= \sum_{W \in \mathcal{G}} b_W L_{WV}\left(\sum_{U \in \mathcal{G}} a_U \chi_U\right) = \sum_{U, W \in \mathcal{G}} b_W a_U \chi_{WVU}.
\end{aligned}$$

Since \mathcal{A} is commutative, it follows that $ST\chi_V = TS\chi_V$, and so $ST = TS$. \square

We now define a natural conjugate \mathcal{A} -linear isomorphism π from \mathcal{M} onto $\mathcal{M}' = \mathcal{N}$ by

$$\pi(A)B\chi_I = BA^*\chi_I, \quad \forall A, B \in \mathcal{M}.$$

In particular, $\pi(A)\chi_I = A^*\chi_I$.

Now we are in a position to prove the parameterization of complete Parseval frame vectors for unitary groups.

THEOREM 3.2. *Let \mathcal{G} be a unitary group on a finitely or countably generated Hilbert \mathcal{A} -module \mathcal{H} over a commutative W^* -algebra \mathcal{A} such that $l_{\mathcal{G}}^2(\mathcal{A})$ is self-dual. Suppose that $\eta \in \mathcal{H}$ be a complete Parseval frame vector for \mathcal{G} . For ξ in \mathcal{H} we have*

- (1) ξ is a complete Parseval frame vector for \mathcal{G} if and only if there exists a unitary operator $A \in \mathcal{G}''$ such that $\xi = A\eta$.
- (2) ξ is a complete frame vector for \mathcal{G} if and only if there exists an invertible and adjointable operator $A \in \mathcal{G}''$ such that $\xi = A\eta$.
- (3) ξ is a complete Bessel sequence vector for \mathcal{G} if and only if there exists an adjointable operator $A \in \mathcal{G}''$ such that $\xi = A\eta$.

PROOF. We will prove (1). The proof of (2) and (3) is similar and we leave it to the interested readers.

By Lemma 2.3, we can assume that $\mathcal{G} = \{L_U|_{\text{Rang}(P)}, U \in \mathcal{G}\}$ and $\eta = P\chi_I$, where P is an orthogonal projection in the commutant of $\{L_U : U \in \mathcal{G}\}$

Let $\mathcal{M} = \{L_U : U \in \mathcal{G}\}''$

First assume that there exists a unitary operator $A \in \mathcal{G}''$ such that $\xi = A\eta$.

We now show that $A\eta$ is a complete Parseval frame vector for \mathcal{G} . For any $x \in \text{Rang}(P)$, we have

$$\begin{aligned}
& \sum_{U \in \mathcal{G}} \langle x, UA\eta \rangle \langle UA\eta, x \rangle = \sum_{U \in \mathcal{G}} \langle x, L_U P A\eta \rangle \langle L_U P A\eta, x \rangle \\
&= \sum_{U \in \mathcal{G}} \langle x, L_U P A P \chi_I \rangle \langle L_U P A P \chi_I, x \rangle = \sum_{U \in \mathcal{G}} \langle x, L_U P A \chi_I \rangle \langle L_U P A \chi_I, x \rangle \\
&= \sum_{U \in \mathcal{G}} \langle x, P L_U A \chi_I \rangle \langle P L_U A \chi_I, x \rangle = \sum_{U \in \mathcal{G}} \langle P x, L_U A \chi_I \rangle \langle L_U A \chi_I, P x \rangle \\
&= \sum_{U \in \mathcal{G}} \langle x, L_U \pi(A^*) \chi_I \rangle \langle L_U \pi(A^*) \chi_I, x \rangle \\
&= \sum_{U \in \mathcal{G}} \langle x, \pi(A^*) L_U \chi_I \rangle \langle \pi(A^*) L_U \chi_I, x \rangle \\
&= \sum_{U \in \mathcal{G}} \langle (\pi(A^*))^* x, \chi_U \rangle \langle \chi_U, (\pi(A^*))^* x \rangle \\
&= \langle (\pi(A^*))^* x, (\pi(A^*))^* x \rangle = \langle x, x \rangle,
\end{aligned}$$

where in the seventh equality we use that fact $\pi(A^*) L_U = L_U \pi(A^*)$, and in the last equality we use that fact that $\pi(A^*)$ is unitary. Therefore $A\eta$ is a complete Parseval frame vector for \mathcal{G} .

Now let $\xi \in \text{Rang}(P)$ be a complete Parseval frame vector for \mathcal{G} . We want to find a unitary operator $A \in \mathcal{G}''$ such that $\xi = A\eta$.

To this aim, we first define an operator $B : l_{\mathcal{G}}^2(\mathcal{A}) \rightarrow l_{\mathcal{G}}^2(\mathcal{A})$ by

$$\chi_U \longmapsto L_U \xi, \quad U \in \mathcal{G}.$$

One can check that B is an adjointable operator and $B^* \chi_V = \sum_{W \in \mathcal{G}} \langle L_{W^{-1}} L_V \eta, \xi \rangle \chi_W$ for any $V \in \mathcal{G}$.

Now for any $U, V \in \mathcal{G}$, we see that

$$\begin{aligned}
& \langle (BB^* - P) \chi_U, \chi_V \rangle \\
&= \langle \sum_{W \in \mathcal{G}} \langle L_{W^{-1}} L_U \eta, \xi \rangle \chi_W, \sum_{S \in \mathcal{G}} \langle L_{S^{-1}} L_V \eta, \xi \rangle \chi_S \rangle - \langle T_\eta U \eta, \chi_V \rangle \\
&= \sum_{W \in \mathcal{G}} \langle L_{W^{-1}} L_U \eta, \xi \rangle \langle \xi, L_{W^{-1}} L_V \eta \rangle - \langle \sum_{W \in \mathcal{G}} \langle U \eta, W \eta \rangle \chi_W, \chi_V \rangle \\
&= \sum_{W \in \mathcal{G}} \langle L_U \eta, L_W \xi \rangle \langle L_W \xi, L_V \eta \rangle - \langle U \eta, V \eta \rangle \\
&= \langle L_U \eta, L_V \eta \rangle - \langle U \eta, V \eta \rangle = \langle L_U P \chi_I, L_V P \chi_I \rangle - \langle U \eta, V \eta \rangle \\
&= \langle L_U T_\eta \eta, L_V T_\eta \eta \rangle - \langle U \eta, V \eta \rangle = \langle T_\eta U \eta, T_\eta V \eta \rangle - \langle U \eta, V \eta \rangle = 0,
\end{aligned}$$

this leads to the fact that $P = BB^*$.

From

$$BL_U\chi_V = B\chi_{UV} = L_{UV}\xi = L_UL_V\xi = L_UB\chi_V,$$

we see that $B \in \mathcal{M}'$. Hence B is a partial isometry in \mathcal{M}' .

Let $Q = B^*B$, then P and Q are equivalent projections in \mathcal{M}' .

Since $l_{\mathcal{G}}^2(\mathcal{A})$ is self-dual, by [17], $End_{\mathcal{A}}^*(l_{\mathcal{G}}^2(\mathcal{A}))$ is a W^* -algebra. Let $(End_{\mathcal{A}}^*(l_{\mathcal{G}}^2(\mathcal{A})))_*$ be its predual. One can check that \mathcal{M} and \mathcal{M}' are $\sigma(End_{\mathcal{A}}^*(l_{\mathcal{G}}^2(\mathcal{A})), (End_{\mathcal{A}}^*(l_{\mathcal{G}}^2(\mathcal{A})))_*)$ -closed in $End_{\mathcal{A}}^*(l_{\mathcal{G}}^2(\mathcal{A}))$, and so both \mathcal{M} and \mathcal{M}' are W^* -algebras (see [20]).

Claim. \mathcal{M} and \mathcal{M}' are finite W^* -algebras.

We now define $\phi : \mathcal{M} \rightarrow \mathcal{A}$ by

$$\phi(A) = \langle A\chi_I, \chi_I \rangle, \quad \forall A \in \mathcal{M}.$$

We want to show that ϕ is a faithful \mathcal{A} -valued trace for \mathcal{M} .

Since $\overline{span\{L_U\chi_I, U \in \mathcal{G}\}} = l_{\mathcal{G}}^2(\mathcal{A})$, for any $A, B \in \mathcal{M}$, we have

$$A\chi_I = \lim_n A_n\chi_I \quad \text{and} \quad B\chi_I = \lim_n B_n\chi_I,$$

where

$$A_n\chi_I = \sum_{i=1}^{k_n} a_i^{(n)} L_{V_i^{(n)}}\chi_I \quad \text{and} \quad B_n\chi_I = \sum_{j=1}^{l_n} b_j^{(n)} L_{W_j^{(n)}}\chi_I$$

for some $a_i^{(n)}, b_j^{(n)} \in \mathcal{A}$ and $V_i^{(n)}, W_j^{(n)} \in \mathcal{G}$.

Then

$$\phi(AB) = \langle AB\chi_I, \chi_I \rangle = \lim_m \lim_n \left\langle \sum_{j=1}^{l_m} \sum_{i=1}^{k_n} b_j^{(m)} a_i^{(n)} L_{W_j^{(m)}} L_{V_i^{(n)}}\chi_I, \chi_I \right\rangle.$$

While

$$\phi(BA) = \lim_n \lim_m \left\langle \sum_{i=1}^{k_n} \sum_{j=1}^{l_m} a_i^{(n)} b_j^{(m)} L_{V_i^{(n)}} L_{W_j^{(m)}}\chi_I, \chi_I \right\rangle.$$

Note that

$$\langle L_{W_j^{(m)}} L_{V_i^{(n)}}\chi_I, \chi_I \rangle = \langle L_{V_i^{(n)}} L_{W_j^{(m)}}\chi_I, \chi_I \rangle.$$

Therefore $\phi(AB) = \phi(BA)$.

If $A \in \mathcal{M}$ is positive and $\phi(A) = 0$, then

$$\langle A^{\frac{1}{2}}\chi_I, A^{\frac{1}{2}}\chi_I \rangle = \langle A\chi_I, \chi_I \rangle = \phi(A) = 0.$$

Thus $A^{\frac{1}{2}}\chi_I = 0$.

Now for any $U \in \mathcal{G}$, we have

$$A^{\frac{1}{2}}\chi_U = A^{\frac{1}{2}}R_U\chi_I = R_U A^{\frac{1}{2}}\chi_I = 0.$$

Therefore $A^{\frac{1}{2}} = 0$, and so $A = 0$. Similarly, by using Lemma 3.1 we can prove that \mathcal{M}' is also finite.

It follows from Proposition 2.4.2 in [20] that $I - P$ and $I - Q$ are equivalent projections in \mathcal{M}' . Therefore there exists a partial isometry $C \in \mathcal{M}'$ such that $CC^* = I - P$ and $C^*C = I - Q$.

Let $T = B + C$. Then T is a unitary operator in \mathcal{M}' , and so $A = (\pi^{-1}(T))^*$ is a unitary operator in \mathcal{M} .

In order to complete the proof it remains to prove that $A\tilde{\eta} = \tilde{\xi}$.

In fact,

$$\begin{aligned} A\eta &= (\pi^{-1}(T))^*P\chi_I = P(\pi^{-1}(T))^*\chi_I \\ &= P\pi(\pi^{-1}(T))\chi_I = PT\chi_I \\ &= P(B + C)\chi_I = PB\chi_I + PC\chi_I \\ &= P\xi = \xi, \end{aligned}$$

which completes the proof. \square

The following result follows immediately from Theorem 3.2 and the fact the set of all the unitary elements in any W^* -algebra is path-connected in norm.

COROLLARY 3.3. *Let \mathcal{G} be a unitary group on a finitely or countably generated Hilbert \mathcal{A} -module \mathcal{H} over a commutative W^* -algebra \mathcal{A} such that $l_{\mathcal{G}}^2(\mathcal{A})$ is self-dual, then the set of all Parseval frame vectors for \mathcal{G} is path-connected.*

4. Parseval Frame Approximations

In the Hilbert space frame setting, the original work on symmetric orthogonalization was done by Löwdin [12] in the late 1970's. The concept of symmetric approximation of frames by Parseval frame was introduced in [5] to extend the symmetric orthogonalization of bases by orthogonal bases in Hilbert spaces. The existence and the uniqueness results for the symmetric approximation of frames by Parseval frames were obtained in [5]. Following their definition, a Parseval frame $\{y_j\}_{j=1}^{\infty}$ is said to be a *symmetric approximation* of frame $\{x_j\}_{j=1}^{\infty}$ in Hilbert space H if it is similar to $\{x_j\}_{j=1}^{\infty}$ and

$$(4.1) \quad \sum_{j=1}^{\infty} \|z_j - x_j\|^2 \geq \sum_{j=1}^{\infty} \|y_j - x_j\|^2$$

is valid for all Parseval frames $\{z_j\}_{j=1}^{\infty}$ of H that are similar to $\{x_j\}_{j=1}^{\infty}$.

Observed by the first author ([6], [7]) that in some situations the symmetric approximation fails to work when the underlying Hilbert

space is infinite dimensional since if we restrict ourselves to the frames induced by a unitary system then the summation in (4.1) is always infinite when the given frame is not Parseval. Instead of using the symmetric approximations to consider the frames generated by a collection of unitary transformations and some window functions, it was proposed to approximate the frame generator by Parseval frame generators. Existence and uniqueness results for such a best approximation were obtained in [6, 7]. We will extend this result to Hilbert C^* -module frames when the underlying C^* -algebra is commutative. It remains open whether this is true when the underlying C^* -algebra is non-commutative.

Following ([7]) we first give the following definition.

DEFINITION 4.1. *Let $\Phi = (\phi_1, \dots, \phi_N)$ be a multi-frame generator for a unitary system \mathcal{U} . Then a Parseval multi-frame generator $\Psi = (\psi_1, \dots, \psi_N)$ for \mathcal{U} is called a best Parseval multi-frame approximation for Φ if the inequality*

$$\sum_{k=1}^N \langle \phi_k - \psi_k, \phi_k - \psi_k \rangle \leq \sum_{k=1}^N \langle \phi_k - \xi_k, \phi_k - \xi_k \rangle$$

is valid for all the Parseval multi-frame generator $\Xi = (\xi_1, \dots, \xi_N)$ for \mathcal{U} .

Let $\Phi = \{\phi_1, \phi_2, \dots, \phi_N\}$ be a multi-frame generator for a unitary system \mathcal{U} on a finitely or countably generated Hilbert \mathcal{A} -module \mathcal{H} over a unital C^* -algebra \mathcal{A} . We use T_Φ to denote the analysis operator from \mathcal{H} to $l_{\mathcal{U} \times \{1, 2, \dots, N\}}^2(\mathcal{A})$ defined by

$$T_\Phi x = \sum_{j=1}^N \sum_{U \in \mathcal{U}} \langle x, U\phi_j \rangle \chi_{(U,j)}, \quad \forall x \in \mathcal{H},$$

where $\{\chi_{(U,j)} : U \in \mathcal{U}, j = 1, 2, \dots, N\}$ is the standard orthonormal basis for $l_{\mathcal{U} \times \{1, 2, \dots, N\}}^2(\mathcal{A})$.

Note that T_Φ is adjointable and its adjoint operator satisfying

$$T_\Phi^* \chi_{(U,j)} = U\phi_j, \quad U \in \mathcal{U}, \quad j = 1, 2, \dots, N.$$

LEMMA 4.2. *Let \mathcal{G} be a unitary group on a Hilbert \mathcal{A} -module \mathcal{H} over a commutative C^* -algebra \mathcal{A} . Suppose that $\Phi = \{\phi_1, \phi_2, \dots, \phi_N\}$ and $\Psi = \{\psi_1, \psi_2, \dots, \psi_N\}$ be two multi-frame generators for \mathcal{G} , then*

$$\sum_{k=1}^N \langle \phi_k, \phi_k \rangle = \sum_{k=1}^N \langle \psi_k, \psi_k \rangle.$$

PROOF. We compute

$$\begin{aligned}
\sum_{k=1}^N \langle \phi_k, \phi_k \rangle &= \sum_{k=1}^N \sum_{j=1}^N \sum_{U \in \mathcal{G}} \langle \phi_k, U\psi_j \rangle \langle U\psi_j, \phi_k \rangle \\
&= \sum_{k=1}^N \sum_{j=1}^N \sum_{U \in \mathcal{G}} \langle U^* \phi_k, \psi_j \rangle \langle \psi_j, U^* \phi_k \rangle \\
&= \sum_{j=1}^N \sum_{k=1}^N \sum_{U \in \mathcal{G}} \langle \psi_j, U^* \phi_k \rangle \langle U^* \phi_k, \psi_j \rangle \\
&= \sum_{j=1}^N \langle \psi_j, \psi_j \rangle.
\end{aligned}$$

□

THEOREM 4.3. *Let \mathcal{G} be a unitary group on a finitely or countably generated Hilbert \mathcal{A} -module \mathcal{H} over a commutative unital C^* -algebra \mathcal{A} . Suppose that $\Phi = \{\phi_1, \phi_2, \dots, \phi_N\}$ is a multi-frame generator for \mathcal{G} . Then $S^{\frac{1}{2}}\Phi$ is the unique best Parseval multi-frame approximation for Φ , where S is the frame operator for the multi-frame $\{U\phi_j : j = 1, \dots, N, U \in \mathcal{G}\}$.*

PROOF. We first show that $S \in \mathcal{G}'$.

For arbitrary $V \in \mathcal{G}$ and $x \in \mathcal{H}$ we have

$$\begin{aligned}
SVx &= \sum_{k=1}^N \sum_{U \in \mathcal{G}} \langle Vx, U\phi_k \rangle U\phi_k \\
&= \sum_{k=1}^N \sum_{U \in \mathcal{G}} \langle x, V^*U\phi_k \rangle U\phi_k \\
&= V \left(\sum_{k=1}^N \sum_{U \in \mathcal{G}} \langle x, V^*U\phi_k \rangle V^*U\phi_k \right) \\
&= V \left(\sum_{k=1}^N \sum_{U \in \mathcal{G}} \langle x, U\phi_k \rangle U\phi_k \right) \\
&= VSx.
\end{aligned}$$

This shows that $S \in \mathcal{G}'$.

Since $\text{End}_{\mathcal{A}}^*(\mathcal{H})$ is a C^* -algebra, by the spectral decomposition for positive elements in C^* -algebra, we can infer that $S^{-\frac{1}{2}}, S^{-\frac{1}{4}} \in \mathcal{G}'$. Therefore $\{S^{-\frac{1}{2}}\phi_1, S^{-\frac{1}{2}}\phi_2, \dots, S^{-\frac{1}{2}}\phi_N\}$ is a complete Parseval multi-frame generator for \mathcal{G} .

Let $\Psi = \{\psi_1, \psi_2, \dots, \psi_N\}$ be any Parseval multi-frame generator for \mathcal{G} . We claim that

$$\sum_{k=1}^N \langle T_{S^{-\frac{1}{2}}\Phi} S^{-\frac{1}{4}}\phi_k, T_{\Psi} S^{-\frac{1}{4}}\phi_k \rangle = \sum_{k=1}^N \langle \psi_k, \phi_k \rangle,$$

where $T_{S^{-\frac{1}{2}}\Phi}$ and T_{Ψ} are the analysis operators with respect to the Parseval multi-frame generators $S^{-\frac{1}{2}}\Phi$ and Ψ respectively.

We compute

$$\begin{aligned} & \sum_{k=1}^N \langle T_{S^{-\frac{1}{2}}\Phi} S^{-\frac{1}{4}}\phi_k, T_{\Psi} S^{-\frac{1}{4}}\phi_k \rangle \\ &= \sum_{k=1}^N \langle \sum_{j=1}^N \sum_{U \in \mathcal{G}} \langle S^{-\frac{1}{4}}\phi_k, U S^{-\frac{1}{2}}\phi_j \rangle \chi_{(U,j)}, \sum_{i=1}^N \sum_{V \in \mathcal{G}} \langle S^{-\frac{1}{4}}\phi_k, V \psi_i \rangle \chi_{(V,i)} \rangle \\ &= \sum_{k=1}^N \sum_{j=1}^N \sum_{U \in \mathcal{G}} \langle S^{-\frac{1}{4}}\phi_k, U S^{-\frac{1}{2}}\phi_j \rangle \langle U \psi_j, S^{-\frac{1}{4}}\phi_k \rangle \\ &= \sum_{k=1}^N \sum_{j=1}^N \sum_{U \in \mathcal{G}} \langle U \psi_j, S^{-\frac{1}{4}}\phi_k \rangle \langle S^{-\frac{1}{4}}\phi_k, U S^{-\frac{1}{2}}\phi_j \rangle \\ &= \sum_{j=1}^N \sum_{k=1}^N \sum_{U \in \mathcal{G}} \langle S^{\frac{1}{4}}\psi_j, U^* S^{-\frac{1}{2}}\phi_k \rangle \langle U^* S^{-\frac{1}{2}}\phi_k, S^{-\frac{1}{4}}\phi_j \rangle \\ &= \sum_{j=1}^N \langle S^{\frac{1}{4}}\psi_j, S^{-\frac{1}{4}}\phi_j \rangle = \sum_{j=1}^N \langle \psi_j, \phi_j \rangle. \end{aligned}$$

We now prove that $S^{\frac{1}{2}}\Phi$ is a best Parseval multi-frame approximation for Φ . We need to show that

$$\sum_{k=1}^N \langle \psi_k - \phi_k, \psi_k - \phi_k \rangle \geq \sum_{k=1}^N \langle S^{-\frac{1}{2}}\phi_k - \phi_k, S^{-\frac{1}{2}}\phi_k - \phi_k \rangle.$$

By Lemma 4.2, it suffices to prove that

$$\sum_{k=1}^N (\langle S^{-\frac{1}{2}}\phi_k, \phi_k \rangle + \langle \phi_k, S^{-\frac{1}{2}}\phi_k \rangle - \langle \psi_k, \phi_k \rangle - \langle \phi_k, \psi_k \rangle) \geq 0.$$

In fact, we have

$$\begin{aligned}
& \sum_{k=1}^N (\langle S^{-\frac{1}{2}} \phi_k, \phi_k \rangle + \langle \phi_k, S^{-\frac{1}{2}} \phi_k \rangle - \langle \psi_k, \phi_k \rangle - \langle \phi_k, \psi_k \rangle) \\
&= \sum_{k=1}^N (\langle S^{-\frac{1}{4}} \phi_k, S^{-\frac{1}{4}} \phi_k \rangle + \langle S^{-\frac{1}{4}} \phi_k, S^{-\frac{1}{4}} \phi_k \rangle) \\
&\quad - \langle T_{S^{-\frac{1}{2}} \Phi} S^{-\frac{1}{4}} \phi_k, T_{\Psi} S^{-\frac{1}{4}} \phi_k \rangle - \langle T_{\Psi} S^{-\frac{1}{4}} \phi_k, T_{S^{-\frac{1}{2}} \Phi} S^{-\frac{1}{4}} \phi_k \rangle) \\
&= \sum_{k=1}^N (\langle T_{S^{-\frac{1}{2}} \Phi} S^{-\frac{1}{4}} \phi_k, T_{S^{-\frac{1}{2}} \Phi} S^{-\frac{1}{4}} \phi_k \rangle + \langle T_{\Psi} S^{-\frac{1}{4}} \phi_k, T_{\Psi} S^{-\frac{1}{4}} \phi_k \rangle) \\
&\quad - \langle T_{S^{-\frac{1}{2}} \Phi} S^{-\frac{1}{4}} \phi_k, T_{\Psi} S^{-\frac{1}{4}} \phi_k \rangle - \langle T_{\Psi} S^{-\frac{1}{4}} \phi_k, T_{S^{-\frac{1}{2}} \Phi} S^{-\frac{1}{4}} \phi_k \rangle) \\
&= \sum_{k=1}^N \langle (T_{S^{-\frac{1}{2}} \Phi} - T_{\Psi}) S^{-\frac{1}{4}} \phi_k, (T_{S^{-\frac{1}{2}} \Phi} - T_{\Psi}) S^{-\frac{1}{4}} \phi_k \rangle \geq 0.
\end{aligned}$$

This implies that $S^{-\frac{1}{2}} \Phi$ is a best Parseval multi-frame approximation for Φ .

For the uniqueness, assume that $\Xi = \{\xi_1, \xi_2, \dots, \xi_N\}$ be another best Parseval multi-frame approximation for Φ . Then we have

$$(4.2) \quad \sum_{k=1}^N \langle \xi_k - \phi_k, \xi_k - \phi_k \rangle = \sum_{k=1}^N \langle S^{-\frac{1}{2}} \phi_k - \phi_k, S^{-\frac{1}{2}} \phi_k - \phi_k \rangle.$$

By Lemma 4.2, we also have

$$(4.3) \quad \sum_{k=1}^N \langle \xi_k, \xi_k \rangle = \sum_{k=1}^N \langle S^{-\frac{1}{2}} \phi_k, S^{-\frac{1}{2}} \phi_k \rangle.$$

Identities (4.2) and (4.3) imply that

$$\begin{aligned}
\sum_{k=1}^N (\langle \xi_k, \phi_k \rangle + \langle \phi_k, \xi_k \rangle) &= \sum_{k=1}^N (\langle S^{-\frac{1}{2}} \phi_k, \phi_k \rangle + \langle \phi_k, S^{-\frac{1}{2}} \phi_k \rangle) \\
&= 2 \sum_{k=1}^N \langle S^{-\frac{1}{4}} \phi_k, S^{-\frac{1}{4}} \phi_k \rangle.
\end{aligned}$$

We claim that

$$\sum_{k=1}^N \langle S^{\frac{1}{4}} \xi_k, S^{\frac{1}{4}} \xi_k \rangle = \sum_{k=1}^N \langle S^{-\frac{1}{4}} \phi_k, S^{-\frac{1}{4}} \phi_k \rangle.$$

In fact,

$$\begin{aligned}
& \sum_{k=1}^N \langle S^{\frac{1}{4}} \xi_k, S^{\frac{1}{4}} \xi_k \rangle \\
&= \sum_{k=1}^N \sum_{j=1}^N \sum_{U \in \mathcal{G}} \langle S^{\frac{1}{4}} \xi_k, U S^{-\frac{1}{2}} \phi_j \rangle \langle U S^{-\frac{1}{2}} \phi_j, S^{\frac{1}{4}} \xi_k \rangle \\
&= \sum_{j=1}^N \sum_{k=1}^N \sum_{U \in \mathcal{G}} \langle S^{-\frac{1}{4}} \phi_j, U^* \xi_k \rangle \langle U^* \xi_k, S^{-\frac{1}{4}} \phi_j \rangle \\
&= \sum_{j=1}^N \langle S^{-\frac{1}{4}} \phi_j, S^{-\frac{1}{4}} \phi_j \rangle.
\end{aligned}$$

Then we have

$$\begin{aligned}
& \sum_{k=1}^N \langle S^{\frac{1}{4}} \xi_k - S^{-\frac{1}{4}} \phi_k, S^{\frac{1}{4}} \xi_k - S^{-\frac{1}{4}} \phi_k \rangle \\
&= \sum_{k=1}^N (\langle S^{\frac{1}{4}} \xi_k, S^{\frac{1}{4}} \xi_k \rangle - \langle S^{\frac{1}{4}} \xi_k, S^{-\frac{1}{4}} \phi_k \rangle \\
&\quad - \langle S^{-\frac{1}{4}} \phi_k, S^{\frac{1}{4}} \xi_k \rangle + \langle S^{-\frac{1}{4}} \phi_k, S^{-\frac{1}{4}} \phi_k \rangle) \\
&= \sum_{k=1}^N (2 \langle S^{-\frac{1}{4}} \phi_k, S^{-\frac{1}{4}} \phi_k \rangle - \langle \xi_k, \phi_k \rangle - \langle \phi_k, \xi_k \rangle) \\
&= 0.
\end{aligned}$$

This implies that

$$S^{\frac{1}{4}} \xi_k = S^{-\frac{1}{4}} \phi_k, \quad k = 1, 2, \dots, N.$$

Therefore

$$\xi_k = S^{-\frac{1}{2}} \phi_k, \quad k = 1, 2, \dots, N.$$

i.e. $\Xi = S^{-\frac{1}{2}} \Phi$, as expected. \square

Acknowledgement. The first author is grateful to Professor M. Frank for many useful communications and help. The authors thank the referee for some valuable suggestions in improving the presentation of the paper.

References

1. X. Dai, D. Larson, Wandering vectors for unitary systems and orthogonal wavelets, *Memoirs Amer. Math. Soc.*, **134** (1998), No. 640.

2. M. Frank, D. Larson, A module frame concept for Hilbert C^* -modules, in *Functional and Harmonic Analysis of wavelets (San Antonio, TX, Jan. 1999)*, *Contemp. Math.*, **247** (1999), 207–233.
3. M. Frank, D. Larson, Modular frames for Hilbert C^* -modules and symmetric approximation of frames, *Proc. SPIE*, **4119** (2000), 325–336.
4. M. Frank, D. Larson, Frames in Hilbert C^* -modules and C^* -algebras, *J. Operator Theory*, **48** (2002), 273–314.
5. M. Frank, V. Paulsen and T. Tiballi, Symmetric approximation of frames and bases in Hilbert spaces, *Trans. Amer. Math. Soc.*, **354** (2002), 777–793.
6. D. Han, Approximation for Gabor and wavelet frames, *Trans. Amer. Math. Soc.*, **355** (2004), 3329–3342.
7. D. Han, Tight frame approximation for multi-frames and super-frames, *J. Approx. Theory*, **129** (2004), 78–93.
8. D. Han, J-P. Gabardo and D. Larson, *Gabor frames and operator algebras, Wavelet Applications in Signal and Image Processing*, *Proc. SPIE.*, **4119** (2000), 337–345.
9. D. Han, D. Larson, Frames, bases and group representations, *Memoirs Amer. Math. Soc.*, **147** (2000), No. 697.
10. G. Kasparov, Hilbert C^* -modules: The theorem of Stinespring and Voiculescu, *J. Operator Theory*, **4** (1980), 133–150.
11. E. Lance, *Hilbert C^* -modules - a Toolkit for Operator Algebraists*, London Mathematical Society lecture Note Series v. 210, Cambridge University Press, Cambridge, England, 1995.
12. P.-O. Löwdin, On the nonorthogonality problem, *Adv. Quantum Chem.*, **5** (1970), 185–199.
13. V. Manuilov, E. Troitsky, *Hilbert C^* -Modules*, American Mathematical Society, Providence, Rhode island, 2005.
14. J. Packer, Applications of the work of Stone and von Neumann to wavelets, *Contemp. Math.*, **365** 2004, 253–279.
15. J. Packer and M. Rieffel, Wavelet filter functions, the matrix completion problem, and projective modules over $C(\mathbb{T}^n)$, *J. Fourier Anal. Appl.*, **9** (2003), 101–116.
16. J. Packer and M. Rieffel, Projective multi-resolution analyses for $L^2(\mathbb{R}^2)$, *J. Fourier Anal. Appl.*, **10** (2004), 439–464.
17. W. Paschke, Inner product modules over B^* -algebras, *Trans. Amer. Math. Soc.*, **182** (1973), 443–468.
18. I. Raeburn, S. Thompson, Countably generated Hilbert modules, the Kasparov stabilization theorem, and frames in Hilbert modules, *Proc. Amer. Math. Soc.*, **131** (2003), 1557–1564.
19. M. Rieffel, Morita equivalence for C^* -algebra and W^* -algebra, *J. Pure Applied Algebra*, **5** (1974), 51–96.
20. S. Sakai, *C^* -algebras and W^* -algebras*, Springer-Verlag, New York, 1998.
21. N. Wegge-Olsen, *K -theory and C^* -algebras - a Friendly Approach*, Oxford University Press, Oxford, England, 1993.
22. P. Wood, Wavelets and Hilbert modules, *J. Fourier Anal. Appl.*, **10** (2004), 573–598.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CENTRAL FLORIDA, ORLANDO, FL 32816

E-mail address: mathjing@gmail.com

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CENTRAL FLORIDA, ORLANDO, FL 32816

E-mail address: dhan@pegasus.cc.ucf.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CENTRAL FLORIDA, ORLANDO, FL 32816

E-mail address: ramm@mail.ucf.edu